Published by Institute of Physics Publishing for SISSA
Received: *May 27, 2008*
Revised: *August 12, 2008*
Accepted: *September 5, 2008*
Published: *September 16, 2008*

JHEP09(2008)075

Moments of heavy quark current correlators at four-loop order in perturbative QCD

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Abstract: We present the result for the first moment of the scalar and axial-vector current correlator in third order of the strong coupling constant α_s and give the details of a recent evaluation of the pseudo-scalar correlator. The results can be used to reduce the theoretical uncertainty due to higher order corrections for the determination of fundamental parameters of QCD in the context of lattice calculations.

Keywords: NLO Computations, Strong Coupling Expansion, QCD.

Contents

1. Introduction

The computation of heavy quark current correlators in perturbation theory is of central interest for many phenomenological applications. For instance their low energy expansion can be related to moments, which have been determined with high precision through the calculation of higher order corrections. Moments of the vector current correlator play an important role in combination with sum rules and the experimentally measured ratio $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. They can be used to perform a precise charm- and bottom-quark mass determination. This method has first been suggested in ref. [1, 2] and has been continuously improved over the years by considering new data from experiment and by including higher order corrections [3 – 5]. However, also the scalar, pseudo-scalar and axial-vector current correlators can be employed in this context. For example, moments of the pseudo-scalar correlator can be used in combination with lattice calculations to determine fundamental constants of QCD, the strong coupling constant α_s and the charm-quark mass with high accuracy [6]. Furthermore, the first eight three-loop moments have been used [7 – 10] as one of several ingredients in order to reconstruct the complete momentum and mass dependence of the scalar, pseudo-scalar, vector and axial-vector polarization functions by means of Padé-approximations. The complete mass and momentum dependence is important in the computation of the Z-boson decays, in which a combination of the vector and axial-vector density enters or in the calculation of Higgs-boson decays which are related to the scalar or pseudo-scalar correlator.

Recently the low energy expansion at three-loop order has been extended and the first 30 coefficients have been determined for the vector current and for the remaining scalar, pseudo-scalar and axial-vector currents in ref. [11, 12], where also singlet contributions have been taken into account. In four-loop order only the first moment of the vector

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current correlator is fully known at present [13, 14]. For double fermionic contributions the first five moments have been computed [15]. Contributions of the order $\alpha_s^j n_l^{j-1}$ have been calculated to all orders j in ref. [16], where n_l denotes the number of light quarks, considered as massless. Numerical results for two moments of the pseudo-scalar correlator were presented in [6].

Whereas the lowest moments of the vector correlator have been studied up to four-loop order already some time ago, the ones of the scalar, pseudo-scalar and axial-vector current correlators were available up to three-loop order only. The purpose of this paper is to provide the still unknown four-loop QCD corrections to the lowest moments of the scalar and axial-vector current correlators and to present the details of the evaluation for the pseudo-scalar correlator, where first numerical results were presented in [6]. Our discussion will be limited to the non-singlet contributions. These results can be useful in combination with lattice calculations to reduce the error due to unknown higher order corrections in perturbation theory in the context of the determination of the strong coupling constant and quark mass as demonstrated in [6]. They can also be seen as a first step towards the evaluation of higher moments. Furthermore analytical results of the pseudo-scalar correlator are presented.

The techniques used in this work have already been successfully applied in several other calculations, among which are the calculation of the matching relation of the strong coupling constant at a heavy quark threshold up to four-loop order in perturbative QCD [17, 18] and the computation of the four-loop QCD corrections to the ρ-parameter arising from top- and bottom-quark loops [19 – 22].

The outline of this paper is as follows: In section 2 we introduce our notations and conventions. In section 3 we discuss the methods of calculation and give the results for the first moment of the scalar and axial-vector current correlators at four-loop order. For completeness we recall the results for the pseudo-scalar and vector case. Finally in section 4 we close with a brief summary and our conclusions. The results for the vector and pseudo-scalar correlator are listed in appendix A, those for the moments with $n = -1$ and $n = 0$ in appendix B.

2. Generalities and notation

The polarization functions for the scalar(s), pseudo-scalar(p), axial-vector(a) and vector(v) current correlator are defined by

$$q^2\,\Pi^\delta(q^2) = i\int dx e^{iqx}\langle 0|Tj^\delta(x)j^\delta(0)|0\rangle, \qquad \text{for } \delta = s, p \tag{2.1}$$

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi^\delta(q^2) + q_\mu q_\nu \Pi_L^\delta(q^2) = i\int dx e^{iqx}\langle 0|Tj_\mu^\delta(x)j_\nu^\delta(0)|0\rangle, \qquad \text{for } \delta = a, v \tag{2.2}$$

with the currents

$$j^s = \overline{\Psi}\Psi, \qquad j^p = i\overline{\Psi}\gamma_5\Psi, \qquad j_\mu^a = \overline{\Psi}\gamma_\mu\gamma_5\Psi, \qquad j_\mu^v = \overline{\Psi}\gamma_\mu\Psi.$$

The low-energy expansion of the polarization functions in $z = q^2/(2\,m)^2$ is conveniently written as

$$\overline{\Pi}^{\delta}(q^2) = \frac{3}{16\,\pi^2} \sum_{n=-1}^{\infty} \overline{C}_n^{\delta}\, \overline{z}^n, \tag{2.3}$$

where the expansion coefficients $\overline{C}_n^{\delta}$ are computed up to four-loop order in perturbative QCD. The expansion in the coupling constant α_s/π up to four-loop order is given by

$$\overline{C}_n^{\delta} = \overline{C}_n^{(0),\delta} + \left(\frac{\alpha_s}{\pi}\right) \overline{C}_n^{(1),\delta} + \left(\frac{\alpha_s}{\pi}\right)^2 \overline{C}_n^{(2),\delta} + \left(\frac{\alpha_s}{\pi}\right)^3 \overline{C}_n^{(3),\delta} + \ldots . \tag{2.4}$$

We decompose the coefficient $\overline{C}_n^{(i),\delta}$ $(i = 0, 1, 2, 3)$ for $n > 0$ into the non-logarithmic and logarithmic parts

$$\overline{C}_n^{(i),\delta} = \sum_{j=0}^{i} \overline{C}_n^{(ij),\delta} l_m^j, \tag{2.5}$$

with $l_m = \log\left(\frac{\overline{m}^2}{\mu^2}\right)$. Furthermore we classify the diagrams with respect to the number of closed fermion loops inserted into a diagram. The symbol n_l denotes the number of light quarks and the symbol $n_h = 1$ denotes a heavy quark with mass m. This decomposition at four-loop order is given by

$$\overline{C}_n^{(3j),\delta} = \overline{C}_{n,0}^{(3j),\delta} + \overline{C}_{n,h}^{(3j),\delta} n_h + \overline{C}_{n,l}^{(3j),\delta} n_l + \overline{C}_{n,hh}^{(3j),\delta} n_h^2 + \overline{C}_{n,hl}^{(3j),\delta} n_h n_l + \overline{C}_{n,ll}^{(3j),\delta} n_l^2 . \tag{2.6}$$

The bar indicates that renormalization of m, α_s and the current has been performed in the $\overline{\text{MS}}$-scheme. We have checked that for $n = 1$ both scalar and axial-vector polarization functions $\overline{\Pi}^{\delta}(q^2)$ obey the standard renormalization group equation(RGE). For the vector current correlator the longitudinal part $\Pi_L^v(q^2)$ of the polarization function is zero due to the vector Ward-identity. The longitudinal part of the axial-vector correlator obeys the axial Ward-identity [23, 24]

$$q^4\, \Pi_L^a(q^2) = (2\,m)^2\, q^2\, \Pi^p(q^2) + \text{contact term}. \tag{2.7}$$

Inserting the expansion of eq. (2.3) leads to

$$\sum_{n=-1}^{\infty} \overline{C}_{L,n}^{a}\, \overline{z}^n = \sum_{n=-1}^{\infty} \overline{C}_n^{p}\, \overline{z}^{n-1} + \frac{1}{q^4} \text{contact term}. \tag{2.8}$$

Performing a shift in the summation index $n = k+1$ on the r.h.s. of eq. (2.8) and comparing the coefficients of the different orders in z^k in both sides leads to

$$\overline{C}_{L,k}^{a} = \overline{C}_{k+1}^{p}, \tag{2.9}$$

for $k \geq -1$, which allows to obtain the second moment of the pseudo-scalar correlator from the calculation of the first moment of the longitudinal part of the axial-vector correlator. The contact term in eq. (2.8) only contributes to the order $1/z^2$.

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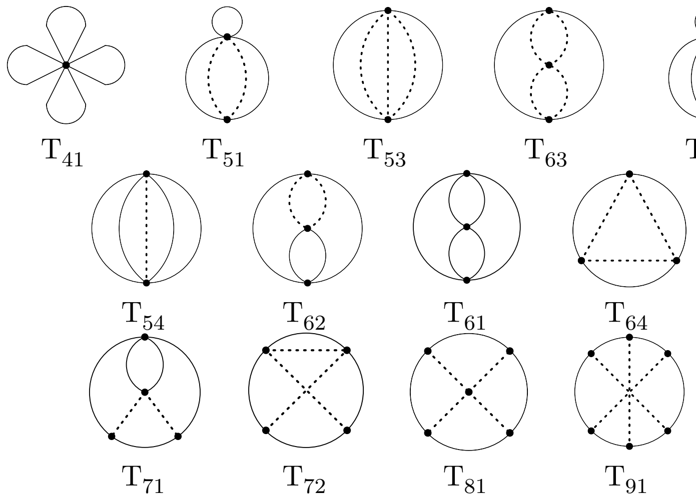

Figure 1: Master integrals in the standard basis. The solid (dashed) lines denote massive (massless) propagators.

3. Calculations and results

Treatment of Feynman integrals

In a first step the program `QGRAF` [25] has been used to generate all necessary diagrams. Subsequently all appearing integrals have been mapped on a small set of 13 master integrals with the traditional Integration-By-Parts(IBP) method [26] in combination with Laporta's algorithm [27, 28]. This procedure has been coded with the help of the programs `FORM` [29 − 31] and `FERMAT` [32]. The remaining master integrals are shown in figure 1 in the standard basis. They have first been determined in ref. [33] with the method of difference equation [34, 28] and subsequently in ref. [35], where the method of ε-finite basis has been introduced. Also other authors [19, 36 − 41] have contributed in this connection with analytical or numerical results.

Renormalization

The calculation has been performed in dimensional regularization [42] with the space-time dimension $d = 4 - 2\,\varepsilon$. The renormalization was accomplished in $\overline{\text{MS}}$-scheme [43, 44]. In order to obtain a finite result for the moments $n \geq 1$, the strong coupling constant and the heavy quark mass require to be renormalized. The bare and renormalized quantities are related through the renormalization constants by

$$m_B = Z_m\,\overline{m}, \qquad \alpha_s^B = Z_{\alpha_s}\,\alpha_s, \tag{3.1}$$

where the index B denotes a bare quantity. In our case the renormalization constant Z_m of the $\overline{\text{MS}}$-mass $\overline{m} = \overline{m}(\mu)$ is needed to order α_s^3 and the renormalization constant Z_{α_s} up to order α_s^2. The corresponding results of the mass anomalous dimension γ_m and the QCD β-function in $\overline{\text{MS}}$-scheme are known since long [45 − 49].

For $n = 0, -1$ the correlators demand an additional additive renormalization [50, 51]. The renormalized polarization functions for the vector and scalar correlator are given by

$$\overline{\Pi}^v(q^2,\alpha_s,\overline{m},\mu) = \mu^{2\varepsilon}\Pi_B^v(q^2,\alpha_s^B,m_B) + \frac{Z_q^{vv}}{16\,\pi^2}, \tag{3.2}$$

$$\overline{\Pi}^s(q^2,\alpha_s,\overline{m},\mu) = \mu^{2\varepsilon}Z_m^2\Pi_B^s(q^2,\alpha_s^B,m_B) + \frac{Z_q^{ss}}{16\,\pi^2} + \frac{1}{z}\,\frac{Z_m^{ss}}{16\,\pi^2}, \tag{3.3}$$

where the factor $\mu^{2\,\varepsilon}$ keeps the mass dimension of the polarization functions independent from ε. In the massless limit the coefficient in front of Z_m^{ss} vanishes, therefore the renormalization constants Z_q^{vv} and Z_q^{ss} can be determined both by the computation of the massless vector and scalar correlator. The renormalization constant Z_q^{vv} is known up to order α_s^3 in refs. [52, 53] and the renormalization constant Z_q^{ss} has been determined in ref. [54] to the same order.

The renormalized axial-vector and pseudo-scalar polarization functions read

$$\overline{\Pi}^a(q^2,\alpha_s,\overline{m},\mu) = \mu^{2\varepsilon}\Pi_B^a(q^2,\alpha_s^B,m_B) + \frac{Z_q^{aa}}{16\,\pi^2} - \frac{1}{z}\,\frac{Z_m^{aa}}{16\,\pi^2}, \tag{3.4}$$

$$\overline{\Pi}_L^a(q^2,\alpha_s,\overline{m},\mu) = \mu^{2\varepsilon}\Pi_{L,B}^a(q^2,\alpha_s^B,m_B) + \frac{1}{z}\,\frac{Z_m^{aa}}{16\,\pi^2}, \tag{3.5}$$

$$\overline{\Pi}^p(q^2,\alpha_s,\overline{m},\mu) = \mu^{2\varepsilon}Z_m^2\Pi_B^p(q^2,\alpha_s^B,m_B) + \frac{Z_q^{pp}}{16\,\pi^2} + \frac{1}{z}\,\frac{Z_m^{pp}}{16\,\pi^2}. \tag{3.6}$$

Some of the renormalization constants are related among each other. For vanishing quark mass the vector and axial-vector correlator as well as the scalar and pseudo-scalar correlator are identical, hence $Z_q^{aa} = Z_q^{vv}$ and $Z_q^{pp} = Z_q^{ss}$. In addition, it was shown in ref. [51] that the use of the axial Ward identity of ref. [23, 24] directly leads to the identity $Z_m^{aa} = Z_q^{pp}$. At last, the renormalization constants Z_m^{ss} and Z_m^{pp} have been computed since long in the course of considering quadratic quark mass corrections (*in the high energy limit*) for the vector [55], scalar and pseudo-scalar [56] correlator.[1]

It should be stressed that, in principle, one does not need to know any additive renormalization constants appearing in eqs. (3.2)–(3.6) for carrying out the renormalization. In fact, these constants could be (and have been) determined from the requirement of the finiteness of the renormalized moments of the corresponding correlators. Thus, the full agreement between the additive renormalization constants obtained in our calculation (based on *massive tadpoles*) and the same constants derived earlier (from considerations dealing with *massless propagators*) provides us with a strong check of our results.

Treatment of γ_5

The pseudo-scalar and axial-vector correlator involve γ_5 in d-dimensions which requires a self-consistent treatment (see e.g. a review [57] and references therein).

In this work we have adopted a naive anti-commuting definition of γ_5. Such a prescription is self-consistent for the non-singlet contributions (for two-point correlators considered in vector-like theories like QCD or QED) which we deal with. Let us shortly outline a proof.[2]

[1] We are grateful to K.G. Chetyrkin for communicating to us these not yet published results.

[2] The below considerations are simple, well-known among experts and not original. We discuss the issue in some detail basically to meet the requirements of the referee.

To be specific, we consider the axial-vector correlator (the pseudo-scalar case represents no extra complications). In this case every contributing Feynman diagram will contain two (and only two because of the assumed vector-likeness of the Lagrangian) γ_5's in *one and the same* fermion loop.[3] Let us consider a particular non-singlet diagram contributing to the axial-vector correlator $\Pi^a_{\mu\nu}$ of eq. (2.2) and write it as follows (loop integrations are understood):

$$F_{\mu\nu} = \mathbf{Tr}[\gamma_\mu\gamma_5\, S_1\, \gamma_\nu\gamma_5\, S_2]\, F_{\rm rest}.$$

Here S_1 and S_2 are numerators of the fermion propagators like

$$S_j = (\not{p}_j + m), \quad j = 1, 2.$$

For simplicity we assume that all fermion propagators depend on only one fermion mass. The denominators and global factors are absorbed in $F_{\rm rest}$. In the case of higher order QCD corrections the symbols S_1, S_2 are understood in a generalized way as products of numerators of the fermion propagators and contain also the QCD vertices. Our γ_5 prescription means that we anti-commutate, say, the left γ_5 to the position next to the right γ_5. Then we use the identity

$$\gamma_5\, \gamma_5 = \mathbb{1}$$

to arrive at a γ_5-free amplitude.

In order to understand why this prescription is self-consistent we should take into account the following two observations.

I.) The main problem with γ_5 in dimensional regularization is that one can not simultaneously meet four basic requirements (see e.g. [58])

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \cdot \mathbb{1}\,, \qquad\qquad g^{\mu}{}_{\mu} = d\,, \tag{3.7}$$

$$\{\gamma_\mu, \gamma_5\} = 0 \text{ for all } \mu \tag{3.8}$$

and

$$\mathbf{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = 4i\epsilon^{\mu\nu\rho\sigma}, \tag{3.9}$$

due to the essentially four-dimensional character of eq. (3.9).

II.) It is rather clear, that if in a particular physical problem no traces like in eq. (3.9) may occur, then *no* problem with γ_5 could appear, which is exactly our case.

The validity of our treatment of γ_5 could also be justified with more formal considerations. Indeed, all experts in the field acknowledge that basically the only rigorous choice for γ_5 in dimensional regularization is based on the original definition by 't Hooft and Veltman [42], formalized by Breitenlohner and Maison [58].

As a consequence of the lost anti-commutativity of γ_5 in such an approach the Ward identities (even the non-anomalous ones) are violated. The way to restore the validity of (non-anomalous) Ward identities was suggested in ref. [59] and elaborated in ref. [46]. It amounts to the introduction of an extra finite renormalization of γ_5-dependent currents.

[3]Diagrams containing each γ_5 in its own fermion loop are, by definition, *singlet* and thus are excluded from our considerations.

A close inspection of ref. [46] clearly demonstrates that for situations like ours (where no objects like eq. (3.9) appear) the finite renormalization *is tuned in such a way*, that it reproduces *exactly* the results obtainable with a completely anti-commutating γ_5.

Results

The four-loop moment with $n = 1$ is known analytically for all four correlators. For the scalar correlator it is given by

$$\begin{aligned}
\overline{C}_{1,0}^{(30),s} &= \frac{240320}{1701}\,a_5 + \frac{513923}{5103}\,a_4 + \frac{32995}{1701}\,\zeta_5 - \frac{1707578737}{3265920}\,\zeta_3 - \frac{6008}{5103}\,\log^5(2) \\
&\quad + \frac{513923}{122472}\,\log^4(2) + \frac{30040}{15309}\,\log^3(2)\,\pi^2 - \frac{513923}{122472}\,\log^2(2)\,\pi^2 \\
&\quad + \frac{20122}{15309}\,\log(2)\,\pi^4 + \frac{11458913}{2939328}\,\pi^4 + \frac{183424051}{4898880}, && (3.10) \\
\overline{C}_{1,h}^{(30),s} &= \frac{13326713}{34020}\,a_4 + \frac{4}{9}\,\zeta_5 + \frac{609136933177}{2514758400}\,\zeta_3 + \frac{13326713}{816480}\,\log^4(2) \\
&\quad - \frac{13326713}{816480}\,\log^2(2)\,\pi^2 - \frac{454390553}{97977600}\,\pi^4 + \frac{91614310199}{3772137600}, && (3.11) \\
\overline{C}_{1,l}^{(30),s} &= \frac{14}{81}\,a_4 + \frac{6278503}{233280}\,\zeta_3 + \frac{7}{972}\,\log^4(2) - \frac{7}{972}\,\log^2(2)\,\pi^2 \\
&\quad - \frac{35927}{116640}\,\pi^4 - \frac{2197597}{349920}, && (3.12) \\
\overline{C}_{1,hh}^{(30),s} &= -\frac{27479}{34020}\,\zeta_3 + \frac{1729337}{1377810}, && (3.13) \\
\overline{C}_{1,hl}^{(30),s} &= -\frac{205}{324}\,a_4 - \frac{38171}{62208}\,\zeta_3 - \frac{205}{7776}\,\log^4(2) + \frac{205}{7776}\,\log^2(2)\,\pi^2 \\
&\quad + \frac{2009}{186624}\,\pi^4 + \frac{1937539}{4199040}, && (3.14) \\
\overline{C}_{1,ll}^{(30),s} &= \frac{7867}{32805}, && (3.15) \\
\overline{C}_{1}^{(31),s} &= \frac{197329}{1296}\,\zeta_3 - \frac{2573}{20} + n_h\left(\frac{2541989}{233280} - \frac{429089}{31104}\,\zeta_3\right) \\
&\quad + n_l\left(\frac{5843}{1215} - \frac{17939}{1944}\,\zeta_3\right) - n_h^2\left(\frac{60559}{349920} - \frac{1435}{5184}\,\zeta_3\right) \\
&\quad + n_l\,n_h\left(\frac{1597}{69984} + \frac{1435}{5184}\,\zeta_3\right) + n_l^2\,\frac{238}{1215}, && (3.16) \\
\overline{C}_{1}^{(32),s} &= \frac{7381}{1620} - n_h\,\frac{671}{1215} - n_l\,\frac{671}{1215} + n_h^2\,\frac{61}{3645} + n_l\,n_h\,\frac{122}{3645} + n_l^2\,\frac{61}{3645}, && (3.17)
\end{aligned}$$

where ζ_n denotes the Riemann zeta-function and $a_n = \mathrm{Li}_n(1/2)$. The result for the axial-vector correlator is given by

$$\begin{aligned}
\overline{C}_{1,0}^{(30),a} &= -\frac{3996704}{25515}\,a_5 + \frac{5966100779}{12247200}\,a_4 + \frac{499588}{382725}\,\log^5(2) \\
&\quad + \frac{5966100779}{293932800}\,\log^4(2) - \frac{499588}{229635}\,\log^3(2)\,\pi^2 - \frac{5966100779}{293932800}\,\log^2(2)\,\pi^2 \\
&\quad - \frac{7151983}{2296350}\,\log(2)\,\pi^4 + \frac{25214645053}{35271936000}\,\pi^4 + \frac{26239187}{34020}\,\zeta_5
\end{aligned}$$

δ	n	$\overline{C}_n^{(0),\delta}$	$\overline{C}_n^{(10),\delta}$	$\overline{C}_n^{(11),\delta}$	$\overline{C}_n^{(20),\delta}$	$\overline{C}_n^{(21),\delta}$	$\overline{C}_n^{(22),\delta}$	$\overline{C}_n^{(30),\delta}$	$\overline{C}_n^{(31),\delta}$	$\overline{C}_n^{(32),\delta}$	$\overline{C}_n^{(33),\delta}$
s	1	0.8000	0.6025	0.0000	−7.7402	−1.2551	0.0000	−5.4135	32.8284	2.6149	0.0000
a	1	0.5333	0.8461	1.0667	−1.1071	1.6925	−0.0444	−2.4297	5.2981	0.4406	0.0321
p	1	1.3333	3.1111	0.0000	0.1154	−6.4815	0.0000	−1.2224	2.5008	13.5031	0.0000
p	2	0.5333	2.0642	1.0667	7.2362	1.5909	−0.0444	7.0659	−7.5852	0.5505	0.0321
v	1	1.0667	2.5547	2.1333	2.4967	3.3130	−0.0889	−5.6404	4.0669	0.9590	0.0642

Table 1: Numerical values for $\overline{C}_1^\delta$ ($\delta = s, a, p, v$) and $\overline{C}_2^p$ for $n_l = 3$ which corresponds to the case of the charmed quark.

$$-\frac{1910114229901}{1306368000}\,\zeta_3 + \frac{6038304844519}{5878656000}, \tag{3.18}$$

$$\overline{C}^{(30),a}_{1,h} = \frac{453463328653}{4115059200} + \frac{920203769}{680400}\,a_4 + \frac{920203769}{16329600}\,\log^4(2) - \frac{182}{27}\,\zeta_5$$
$$-\frac{920203769}{16329600}\,\log^2(2)\,\pi^2 - \frac{6237707419}{391910400}\,\pi^4 + \frac{2277007338343}{2743372800}\,\zeta_3, \tag{3.19}$$

$$\overline{C}^{(30),a}_{1,l} = -\frac{33628673}{839808} - \frac{12907}{29160}\,a_4 - \frac{12907}{699840}\,\log^4(2) + \frac{12907}{699840}\,\log^2(2)\,\pi^2$$
$$-\frac{16384897}{16796160}\,\pi^4 + \frac{313282529}{2799360}\,\zeta_3, \tag{3.20}$$

$$\overline{C}^{(30),a}_{1,hh} = \frac{15134719}{16533720} - \frac{303799}{408240}\,\zeta_3, \tag{3.21}$$

$$\overline{C}^{(30),a}_{1,hl} = \frac{21592349}{50388480} - \frac{1499}{3888}\,a_4 - \frac{1499}{93312}\,\log^4(2) + \frac{1499}{93312}\,\log^2(2)\,\pi^2$$
$$+\frac{73451}{11197440}\,\pi^4 - \frac{2539913}{3732480}\,\zeta_3, \tag{3.22}$$

$$\overline{C}^{(30),a}_{1,ll} = \frac{42133}{196830} - \frac{56}{405}\,\zeta_3, \tag{3.23}$$

$$\overline{C}^{(31),a}_{1} = \frac{96167813}{311040}\,\zeta_3 - \frac{164649889}{466560} - n_h^2\left(\frac{806681}{4199040} - \frac{10493}{62208}\,\zeta_3\right)$$
$$-n_l\,n_h\left(\frac{349337}{4199040} - \frac{10493}{62208}\,\zeta_3\right) + n_l^2\,\frac{397}{3645} + n_h\left(\frac{33280729}{933120}\right.$$
$$\left.-\frac{59917211}{1866240}\,\zeta_3\right) + n_l\left(\frac{22748503}{699840} - \frac{14152979}{466560}\,\zeta_3\right), \tag{3.24}$$

$$\overline{C}^{(32),a}_{1} = -\frac{1819}{1620} + n_h\,\frac{1442}{3645} + n_l\,\frac{1442}{3645} - n_h^2\,\frac{13}{10935}$$
$$-n_l\,n_h\,\frac{26}{10935} - n_l^2\,\frac{13}{10935}, \tag{3.25}$$

$$\overline{C}^{(33),a}_{1} = \frac{7}{15} - n_h\,\frac{4}{27} - n_l\,\frac{4}{27} + n_h^2\,\frac{4}{405} + n_l\,n_h\,\frac{8}{405} + n_l^2\,\frac{4}{405}. \tag{3.26}$$

Numerical results for the pseudo-scalar case have first been presented in ref. [6]. The analytical result is given in appendix A, together with the corresponding one for the vector current correlator, taken from ref. [13, 14]. The coefficients $\overline{C}_1^\delta$ ($\delta = s, a, p, v$) and $\overline{C}_2^p$ are listed in numerical form in table 1. In appendix B we provide in addition the expansion coefficients for $n = 0, -1$. Except for $\overline{C}_0^a$ they are only known numerically, however, with high precision. A completely analytical result would require the analytical determination of

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the constants $T_{54,2}$, $T_{64,2}$, $T_{61,2}$, $T_{62,3}$, $T_{72,1}$, $T_{71,1}$, $T_{81,1}$, $T_{91,1}$, where $T_{n,i}$ are the coefficients of the ε-expansion of the master integrals T_n as defined in figure 1

$$T_n = \sum_{i=n_{\min}}^{\infty} \varepsilon^i T_{n,i}\,. \tag{3.27}$$

4. Summary and conclusion

We have computed the lowest coefficients in the low-energy expansion of the scalar and axial-vector current correlators at four-loop order in perturbative QCD. All appearing loop-integrals have been reduced to known master integrals, using Laporta's algorithm. We also gave the details of the calculation as well as the analytical results for the moments of the pseudo-scalar correlator. These results allow to reduce the theoretical error originating from higher order corrections in the determination of fundamental constants of QCD, like the strong coupling constant and the charm-quark mass in the context of lattice calculations. Our results are also available in computer readable form under the URL `http://arxiv.org` by downloading the source of this article.

Acknowledgments

C.S. would like to thank K.G. Chetyrkin, J.H. Kühn and M. Steinhauser for their support and many discussions. This research was supported by U.S. Department of Energy contract No.DE-AC02-98CH10886.

A. Moments for the pseudo-scalar and vector correlator

The analytical result for the first two moments of the pseudo-scalar correlator presented in numerical from in ref. [6] is given by

$$\begin{aligned}
\overline{C}^{(30),p}_{1,0} &= \frac{10304}{243}\,a_5 - \frac{130535}{1458}\,a_4 - \frac{35189}{243}\,\zeta_5 + \frac{4653637}{15552}\,\zeta_3 - \frac{1288}{3645}\,\log^5(2) \\
&\quad - \frac{130535}{34992}\,\log^4(2) + \frac{1288}{2187}\,\log^3(2)\,\pi^2 + \frac{130535}{34992}\,\log^2(2)\,\pi^2 \\
&\quad + \frac{10094}{10935}\,\log(2)\,\pi^4 - \frac{5686729}{4199040}\,\pi^4 - \frac{3732431}{34992}, \qquad (A.1)\\
\overline{C}^{(30),p}_{1,h} &= -\frac{294727}{2430}\,a_4 + \frac{80}{9}\,\zeta_5 - \frac{7656133}{85050}\,\zeta_3 - \frac{294727}{58320}\,\log^4(2) \\
&\quad + \frac{294727}{58320}\,\log^2(2)\,\pi^2 + \frac{9593011}{6998400}\,\pi^4 - \frac{373843}{453600}, \qquad (A.2)\\
\overline{C}^{(30),p}_{1,l} &= \frac{115}{243}\,a_4 - \frac{157783}{7776}\,\zeta_3 + \frac{115}{5832}\,\log^4(2) - \frac{115}{5832}\,\log^2(2)\,\pi^2 \\
&\quad + \frac{115709}{699840}\,\pi^4 - \frac{7381}{5832}, \qquad (A.3)\\
\overline{C}^{(30),p}_{1,hh} &= \frac{403}{630}\,\zeta_3 - \frac{9493}{25515}, \qquad (A.4)\\
\overline{C}^{(30),p}_{1,hl} &= -\frac{4}{9}\,a_4 - \frac{1}{54}\,\log^4(2) + \frac{1}{54}\,\log^2(2)\,\pi^2 + \frac{49}{6480}\,\pi^4 + \frac{35}{96}\,\zeta_3 - \frac{3457}{11664}, \qquad (A.5)
\end{aligned}$$

$$\overline{C}_{1,ll}^{(30),p} = \frac{277}{729}, \tag{A.6}$$

$$\overline{C}_{1}^{(31),p} = -\frac{71203}{864}\zeta_3 + \frac{143465}{1296} - n_h\left(\frac{12439}{1944} - \frac{2315}{1296}\zeta_3\right) - n_l\left(\frac{17191}{1944} - \frac{6473}{1296}\zeta_3\right) + n_h^2\left(\frac{14}{243} + \frac{7}{36}\zeta_3\right) + n_l\,n_h\left(\frac{64}{243} + \frac{7}{36}\zeta_3\right) + n_l^2\,\frac{50}{243}, \tag{A.7}$$

$$\overline{C}_{1}^{(32),p} = \frac{847}{36} - n_h\,\frac{77}{27} - n_l\,\frac{77}{27} + n_h^2\,\frac{7}{81} + n_l\,n_h\,\frac{14}{81} + n_l^2\,\frac{7}{81}, \tag{A.8}$$

$$\begin{aligned}\overline{C}_{2,0}^{(30),p} = {} & \frac{278048}{945}\,a_5 - \frac{3509250197}{4082400}\,a_4 - \frac{45178393}{34020}\,\zeta_5 + \frac{2871407869129}{1306368000}\,\zeta_3 \\ & -\frac{34756}{14175}\,\log^5(2) - \frac{3509250197}{97977600}\,\log^4(2) + \frac{3509250197}{97977600}\,\log^2(2)\,\pi^2 \\ & +\frac{34756}{8505}\,\log^3(2)\,\pi^2 + \frac{180277}{28350}\,\log(2)\,\pi^4 - \frac{94380515779}{11757312000}\,\pi^4 \\ & -\frac{509351043139}{653184000},\end{aligned} \tag{A.9}$$

$$\begin{aligned}\overline{C}_{2,h}^{(30),p} = {} & -\frac{978527581}{680400}\,a_4 + \frac{62}{3}\,\zeta_5 - \frac{8948001289387}{10059033600}\,\zeta_3 - \frac{978527581}{16329600}\,\log^4(2) \\ & +\frac{978527581}{16329600}\,\log^2(2)\,\pi^2 + \frac{6586806599}{391910400}\,\pi^4 - \frac{351736938533}{3017710080},\end{aligned} \tag{A.10}$$

$$\begin{aligned}\overline{C}_{2,l}^{(30),p} = {} & -\frac{493}{3240}\,a_4 - \frac{151413217}{933120}\,\zeta_3 - \frac{493}{77760}\,\log^4(2) + \frac{493}{77760}\,\log^2(2)\,\pi^2 \\ & +\frac{1044179}{622080}\,\pi^4 + \frac{8402929}{279936},\end{aligned} \tag{A.11}$$

$$\overline{C}_{2,hh}^{(30),p} = \frac{36809}{136080}\,\zeta_3 - \frac{1699529}{5511240}, \tag{A.12}$$

$$\begin{aligned}\overline{C}_{2,hl}^{(30),p} = {} & -\frac{179}{1296}\,a_4 + \frac{17839}{1244160}\,\zeta_3 - \frac{179}{31104}\,\log^4(2) \\ & +\frac{179}{31104}\,\log^2(2)\,\pi^2 + \frac{8771}{3732480}\,\pi^4 - \frac{1951867}{16796160},\end{aligned} \tag{A.13}$$

$$\overline{C}_{2,ll}^{(30),p} = -\frac{56}{405}\,\zeta_3 + \frac{15511}{65610}, \tag{A.14}$$

$$\begin{aligned}\overline{C}_{2}^{(31),p} = {} & -\frac{54646039}{103680}\,\zeta_3 + \frac{97431227}{155520} - n_h\left(\frac{18258607}{311040} - \frac{30278653}{622080}\,\zeta_3\right) \\ & -n_l\left(\frac{13928429}{233280} - \frac{7668337}{155520}\,\zeta_3\right) - n_h^2\left(\frac{39137}{1399680} - \frac{1253}{20736}\,\zeta_3\right) \\ & +n_h\,n_l\left(\frac{55711}{1399680} + \frac{1253}{20736}\,\zeta_3\right) + n_l^2\,\frac{247}{3645},\end{aligned} \tag{A.15}$$

$$\overline{C}_{2}^{(32),p} = \frac{257}{540} - n_h\,\frac{136}{1215} - n_l\,\frac{136}{1215} + n_h^2\,\frac{119}{3645} + n_h\,n_l\,\frac{238}{3645} + n_l^2\,\frac{119}{3645}, \tag{A.16}$$

$$\overline{C}_{2}^{(33),p} = \frac{7}{15} - n_h\,\frac{4}{27} - n_l\,\frac{4}{27} + n_h^2\,\frac{4}{405} + n_h\,n_l\,\frac{8}{405} + n_l^2\,\frac{4}{405}. \tag{A.17}$$

The first moment of the vector current correlator has been obtained in ref. [13, 14]

$$\begin{aligned}\overline{C}_{1,0}^{(30),v} = {} & -\frac{1019840}{5103}\,a_5 - \frac{84951877}{306180}\,a_4 - \frac{3655}{10206}\,\zeta_5 + \frac{17554601717}{32659200}\,\zeta_3 + \frac{25496}{15309}\,\log^5(2) \\ & -\frac{84951877}{7348320}\,\log^4(2) - \frac{127480}{45927}\,\log^3(2)\,\pi^2 + \frac{84951877}{7348320}\,\log^2(2)\,\pi^2\end{aligned}$$

$$-\frac{359687}{229635}\log(2)\,\pi^4-\frac{2653167371}{881798400}\,\pi^4-\frac{5397779543}{146966400}, \tag{A.18}$$

$$\overline{C}^{(30),v}_{1,h} = -\frac{1394804}{8505}\,a_4+\frac{128}{27}\,\zeta_5-\frac{95617883401}{943034400}\,\zeta_3-\frac{348701}{51030}\,\log^4(2)$$
$$+\frac{348701}{51030}\,\log^2(2)\,\pi^2+\frac{1447057}{765450}\,\pi^4-\frac{27670774337}{1414551600}, \tag{A.19}$$

$$\overline{C}^{(30),v}_{1,l} = -\frac{4793}{7290}\,a_4-\frac{48350497}{1399680}\,\zeta_3-\frac{4793}{174960}\,\log^4(2)$$
$$+\frac{4793}{174960}\,\log^2(2)\,\pi^2+\frac{372689}{839808}\,\pi^4-\frac{9338899}{2099520}, \tag{A.20}$$

$$\overline{C}^{(30),v}_{1,hh} = -\frac{3287}{7290}\,\zeta_3+\frac{163868}{295245}, \tag{A.21}$$

$$\overline{C}^{(30),v}_{1,hl} = -\frac{116}{243}\,a_4-\frac{38909}{58320}\,\zeta_3-\frac{29}{1458}\,\log^4(2)$$
$$+\frac{1421}{174960}\,\pi^4+\frac{29}{1458}\,\log^2(2)\,\pi^2+\frac{262877}{787320}, \tag{A.22}$$

$$\overline{C}^{(30),v}_{1,ll} = \frac{42173}{98415}-\frac{112}{405}\,\zeta_3, \tag{A.23}$$

$$\overline{C}^{(31),v}_{1} = \frac{7236859}{38880}-\frac{10589033}{77760}\,\zeta_3-n_h\left(\frac{520823}{34992}-\frac{1049579}{116640}\,\zeta_3\right)$$
$$-n_l\left(\frac{1103117}{58320}-\frac{1305359}{116640}\,\zeta_3\right)-n_h^2\left(\frac{14483}{65610}-\frac{203}{972}\,\zeta_3\right)$$
$$-n_h\,n_l\left(\frac{3779}{65610}-\frac{203}{972}\,\zeta_3\right)+n_l^2\,\frac{1784}{10935}, \tag{A.24}$$

$$\overline{C}^{(32),v}_{1} = -\frac{451}{405}+n_h\,\frac{1574}{3645}+n_l\,\frac{1574}{3645}+n_h^2\,\frac{236}{10935}+n_h\,n_l\,\frac{472}{10935}+n_l^2\,\frac{236}{10935}, \tag{A.25}$$

$$\overline{C}^{(33),v}_{1} = \frac{14}{15}-n_h\,\frac{8}{27}-n_l\,\frac{8}{27}+n_h^2\,\frac{8}{405}+n_h\,n_l\,\frac{16}{405}+n_l^2\,\frac{8}{405}. \tag{A.26}$$

B. Moments for $\mathbf{n = -1, 0}$

The moments for $n = -1$ and $n = 0$ exhibit an overall UV divergence. The corresponding $1/\varepsilon$-poles are dropped by definition in the $\overline{\mathrm{MS}}$-scheme. The finite parts are given by

$$\overline{C}^{(3),s}_{-1} = -325.6276432+16.39537650\,n_h+19.76434509\,n_l$$
$$-1.670198265\,n_h^2-0.9856898698\,n_l\,n_h+0.7103788267\,n_l^2, \tag{B.1}$$

$$\overline{C}^{(3),s}_{0} = -82.24477459+14.55555905\,n_h+22.86448444\,n_l$$
$$-0.6046449347\,n_h^2-1.620853371\,n_l\,n_h-1.047986065\,n_l^2, \tag{B.2}$$

$$\overline{C}^{(3),p}_{-1} = -358.5973626+30.02655475\,n_h+35.65136262\,n_l$$
$$-1.485773643\,n_h^2-2.003379176\,n_l\,n_h-0.2645810947\,n_l^2, \tag{B.3}$$

$$\overline{C}^{(3),p}_{0} = -25.62696915+8.147150256\,n_h+6.938402913\,n_l$$
$$+0.08246295965\,n_h^2-0.5283433562\,n_l\,n_h-0.4972293123\,n_l^2, \tag{B.4}$$

$$\overline{C}^{(3),a}_{-1} = 25.62696915-8.147150256\,n_h-6.938402913\,n_l$$
$$-0.08246295965\,n_h^2+0.5283433562\,n_l\,n_h+0.4972293123\,n_l^2, \tag{B.5}$$

$$\overline{\mathcal{C}}_{0,0}^{(30),a} = -\frac{59584}{729}\,a_5 + \frac{362533}{4374}\,a_4 + \frac{668057}{5832}\,\zeta_5 - \frac{29132419}{233280}\,\zeta_3 + \frac{7448}{10935}\,\log^5(2) + \frac{362533}{104976}\,\log^4(2) - \frac{7448}{6561}\,\log^3(2)\,\pi^2 - \frac{362533}{104976}\,\log^2(2)\,\pi^2 - \frac{35584}{32805}\,\log(2)\,\pi^4 - \frac{309377}{2519424}\,\pi^4 + \frac{402928129}{4199040}, \tag{B.6}$$

$$\overline{\mathcal{C}}_{0,ll}^{(30),a} = -\frac{31}{162}\,\zeta_3 - \frac{1655}{69984}, \tag{B.7}$$

$$\overline{\mathcal{C}}_{0,hl}^{(30),a} = \frac{2}{27}\,a_4 - \frac{1919}{2592}\,\zeta_3 + \frac{1}{324}\,\log^4(2) - \frac{1}{324}\,\log^2(2)\,\pi^2 - \frac{49}{38880}\,\pi^4 + \frac{4255}{8748}, \tag{B.8}$$

$$\overline{\mathcal{C}}_{0,hh}^{(30),a} = -\frac{913}{1134}\,\zeta_3 + \frac{317311}{489888}, \tag{B.9}$$

$$\overline{\mathcal{C}}_{0,l}^{(30),a} = -\frac{1025}{729}\,a_4 + \frac{310001}{23328}\,\zeta_3 - \frac{1025}{17496}\,\log^4(2) + \frac{1025}{17496}\,\log^2(2)\,\pi^2 - \frac{18635}{419904}\,\pi^4 - \frac{339551}{139968}, \tag{B.10}$$

$$\overline{\mathcal{C}}_{0,h}^{(30),a} = \frac{859259}{7290}\,a_4 - \frac{85}{27}\,\zeta_5 + \frac{12520907}{145800}\,\zeta_3 + \frac{859259}{174960}\,\log^4(2) - \frac{859259}{174960}\,\log^2(2)\,\pi^2 - \frac{28913567}{20995200}\,\pi^4 - \frac{556867}{1166400}, \tag{B.11}$$

where for compactness $\mu = \overline{m}$ has been chosen. The logarithms $\log(\mu^2/\overline{m}^2)$ can be reconstructed with the help of the RGE.

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